

# Quantum points/patterns, Part 2.

## From quantum points to quantum patterns via multiresolution

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### ABSTRACT

It is obvious that we still have not any unified framework covering a zoo of interpretations of Quantum Mechanics, as well as satisfactory understanding of main ingredients of a phenomena like entanglement. The starting point is an idea to describe properly the key ingredient of the area, namely point/particle-like objects (physical quantum points/particles or, at least, structureless but quantum objects) and to change point (wave) functions by sheaves to the sheaf wave functions (Quantum Sheaves). In such an approach Quantum States are sections of the coherent sheaves or contravariant functors from the kinematical category describing space-time to other one, Quantum Dynamical Category, properly describing the complex dynamics of Quantum Patterns. The objects of this category are some filtrations on the functional realization of Hilbert space of Quantum States. In this Part 2, the sequel of Part 1, we present a family of methods which can describe important details of complex behaviour in quantum ensembles: the creation of nontrivial patterns, localized, chaotic, entangled or decoherent, from the fundamental basic localized (nonlinear) eigenmodes (in contrast with orthodox gaussian-like) in various collective models arising from the quantum hierarchies described by Wigner-like equations.

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### ABSTRACT

It is obvious that we still have not any unified framework covering a zoo of interpretations of Quantum Mechanics, as well as satisfactory understanding of main ingredients of a phenomena like entanglement. The starting point is an idea to describe properly the key ingredient of the area, namely point/particle-like objects (physical quantum points/particles or, at least, structureless but quantum objects) and to change point (wave) functions by sheaves to the sheaf wave functions (Quantum Sheaves). In such an approach Quantum States are sections of the coherent sheaves or contravariant functors from the kinematical category describing space-time to other one, Quantum Dynamical Category, properly describing the complex dynamics of Quantum Patterns. The objects of this category are some filtrations on the functional realization of Hilbert space of Quantum States. In this Part 2, the sequel of Part 1, we present a family of methods which can describe important details of complex behaviour in quantum ensembles: the creation of nontrivial patterns, localized, chaotic, entangled or decoherent, from the fundamental basic localized (nonlinear) eigenmodes (in contrast with orthodox gaussian-like) in various collective models arising from the quantum hierarchies described by Wigner-like equations.

**Keywords:** Localization; quantum states; multiscales; hidden symmetry; sheaves.

### 1. NEW LOCALIZED MODES AND PATTERNS: WHY NEED WE THEM?

It is widely known that the currently available experimental techniques in the area of quantum physics as well as the present level of the understanding of phenomenological models, outstrips the actual level of mathematical description [1], [2]. Considering the problem of describing the really existing and/or realizable states, one should not expect that well-known trivial states like (gaussian) coherent states, harmonic plane waves or eigenstates of orthodox (quantum) Hamiltonians would be enough to characterize the power of complex quantum phenomena. The complexity of a set of relevant states, including entangled (chaotic) ones is still far from being clearly understood and moreover from being realizable [3], [4].

Our motivations arise from the following general questions:

how can we represent a well localized and reasonable state in mathematically correct form?

is it possible to create entangled and other relevant states by means of these new localized building blocks?

The general idea is rather simple: it is well known that the generating symmetry is the key ingredient of any modern reasonable physical theory. Roughly speaking, the representation theory of the underlying (internal/hidden) symmetry (classical or quantum, finite or infinite dimensional, continuous or discrete) is the useful instrument for the description of (orbital) dynamics.

The proper representation theory is well known as “local nonlinear harmonic analysis”, in particular case of the simple underlying symmetry, affine group, aka wavelet analysis [5]–[7]. From our point of view the advantages of such approach are as follows:

- i) the natural realization of localized states in any proper functional realization of (Hilbert) space of states,
- ii) the hidden symmetry of a chosen realization of the functional model describes the (whole) spectrum of possible states via the so-called multiresolution technique providing the exact multiscale decomposition.

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Effects we are interested in are as follows:

- 1). a hierarchy of internal/hidden scales (time, space, phase space);
- 2). non-perturbative multiscales: from slow to fast contributions, from the coarser to the finer level of resolution/decomposition;
- 3). the coexistence of the levels of hierarchy of multiscale dynamics with transitions/intermittency between scales;
- 4). the realization of the key features of the complex quantum world such as the existence of chaotic and/or entangled states with possible destruction in “open/dissipative” regimes due to interactions with quantum/classical environment and transition to decoherent states.

At this level, we may interpret the effect of mysterious entanglement or “quantum interaction” as a result of the simple interscale interaction or intermittency (with allusion to hydrodynamics), i.e. the mixing of orbits generated by multiresolution representation of the hidden underlying symmetry. Surely, the existence of such a symmetry is a natural physical property of the model as well as the structure/type of the space of representation and its proper functional realization. So, instantaneous quantum interaction materializes not in the physical space-time variety but in the space of the representation of hidden symmetry along the orbits/scales constructed by proper representations. Such an approach provides the explicit analytical construction for solutions of c- and q-hierarchies and their important reductions starting from the quantization of c-BBGKY hierarchy [8]–[22]. It is based on tensor algebra extensions of multiresolution representation [5] for states and observables and variational formulation [8]–[22]. We provide the explicit representation for the hierarchy of n-particle reduced distribution functions in the base of the high-localized generalized coherent (regarding underlying generic symmetry (affine group in the simplest case)) states given by the polynomial tensor algebra of proper (in exact sense) basis functions (wavelet families, wavelet packets [6]–[7]), which takes into account contributions from all underlying hidden multiscales from the coarsest scale of resolution to the finest one to provide the full information about (quantum) dynamical process. The difference between classical and quantum case is concentrated in the structure of the set of operators and proper functional spaces where they are realized, included in the set-up, and, of course, depends on the method of quantization. But, in the naive Wigner-Weyl approach for the quantum case, the symbols of operators play the same role as usual functions in the classical case. In some sense, our approach for ensembles (hierarchies) resembles Bogolyubov’s one and related approaches but we do not use any perturbation technique (like virial expansion) or linearization procedures. Most important, that numerical modeling in all cases shows the creation of various internal (coherent) structures from localized modes, which are related to the (meta)stable (equilibrium) or unstable type of behaviour and corresponding patterns (waveletons) formation [8]–[22].

We start from the second quantized representation for an algebra of observables  $A = (A_0, A_1, \dots, A_s, \dots)$  in the standard form

$$A = A_0 + \int dx_1 \Psi^+(x_1) A_1 \Psi(x_1) + \dots \\ + (s!)^{-1} \int dx_1 \dots dx_s \Psi^+(x_1) \dots \Psi^+(x_s) A_s \Psi(x_s) \dots \Psi(x_1) + \dots$$

N-particle Wigner functions allow to consider them as partitions representing some useful quasiprobabilities. The full description for quantum ensemble can be done by the hierarchy of functions (symbols):

$$W = \{W_s(x_1, \dots, x_s), s = 0, 1, 2 \dots\},$$

which are solutions of Wigner (pseudodifferential) equations:

$$\frac{\partial W_n}{\partial t} = -\frac{p}{m} \frac{\partial W_n}{\partial q} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\hbar/2)^{2\ell}}{(2\ell+1)!} \frac{\partial^{2\ell+1} U_n(q)}{\partial q^{2\ell+1}} \frac{\partial^{2\ell+1} W_n}{\partial p^{2\ell+1}}. \quad (1)$$

The similar Lindblad equations describe the important decoherence processes [4].

## 2. VARIATIONAL MULTIREOLUTION REPRESENTATION

We obtain our multiscale/multiresolution representations for solutions of Wigner-like equations (1) via the variational–multiresolution approach. We represent the solutions as decomposition into localized eigenmodes related to a underlying set of scales corresponding to proper orbits generated by action of hidden internal symmetry, like (non-abelian) affine group in the simplest but important case of wavelet analysis:

$$W_n(t, q, p) = \bigoplus_{i=i_c}^{\infty} W_n^i(t, q, p),$$

where value  $i_c$  corresponds to the coarsest level of resolution  $c$  in the full Multiresolution Analysis Decomposition (MRA) of the underlying functional space [5]:

$$V_c \subset V_{c+1} \subset V_{c+2} \subset \dots$$

and  $p = (p_1, p_2, \dots)$ ,  $q = (q_1, q_2, \dots)$ ,  $x_i = (p_1, q_1, \dots, p_i, q_i)$  are coordinates in phase space. We introduce the Fock-like space structure on the whole space of internal hidden scales

$$H = \bigoplus_i \bigotimes_n H_i^n$$

for the set of  $n$ -partial Wigner functions (states):

$$W^i = \{W_0^i, W_1^i(x_1; t), \dots, W_N^i(x_1, \dots, x_N; t), \dots\},$$

where  $W_p(x_1, \dots, x_p; t) \in H^p$ ,  $H^0 = C$ ,  $H^p = L^2(R^{6p})$  or any different proper functional space with the natural Fock space like norm:

$$(W, W) = W_0^2 + \sum_i \int W_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^i \mu_{\ell}.$$

First of all, we consider  $W = W(t)$  as a function of time only,  $W \in L^2(R)$ , via multiresolution decomposition which naturally and efficiently introduces an infinite sequence of the underlying hidden scales [5]. We have the contribution to the final result from each scale of resolution from the whole infinite scale of spaces or more correctly mathematically, from the Tower of Filtration. The closed subspace  $V_j (j \in \mathbf{Z})$  corresponds to the level  $j$  of resolution and satisfies the following properties: let  $D_j$  be the orthonormal complement of  $V_j$  with respect to  $V_{j+1}$ :  $V_{j+1} = V_j \oplus D_j$ . Then we have the following decomposition:

$$\{W(t)\} = \bigoplus_{-\infty < j < \infty} D_j = \overline{V_c \bigoplus_{j=0}^{\infty} D_j},$$

in case when  $V_c$  is the coarsest scale of resolution. The subgroup of translations generates a basis for the fixed scale number:  $\text{span}_{k \in \mathbf{Z}} \{2^{j/2} \Psi(2^j t - k)\} = D_j$ . The whole basis is generated by the action of the full affine group:

$$\text{span}_{k \in \mathbf{Z}, j \in \mathbf{Z}} \{2^{j/2} \Psi(2^j t - k)\} = \text{span}_{k, j \in \mathbf{Z}} \{\Psi_{j, k}\} = \{W(t)\}.$$

After the construction of the multidimensional tensor product bases [4], the next key point is the so-called Fast Wavelet Transform (FWT) [5]–[7], demonstrating that for a large class of operators the wavelet functions are a good approximation for true eigenvectors and the corresponding matrices are almost diagonal. We have the simple linear parametrization of the matrix representation of our operators in the localized wavelet bases and of the action of these operators on arbitrary vectors/states in the proper functional space. FWT provides the maximum sparse and useful form for the wide classes of operators [5]–[7]. After that, we can obtain our multiscale/multiresolution representations for observables (symbols), states, partitions via the variational approaches.

Let  $L$  be an arbitrary (non)linear differential/integral operator with matrix dimension  $d$  (finite or infinite), which acts on some set of functions from  $L^2(\Omega^{\otimes n})$ :  $\Psi \equiv \Psi(t, x_1, x_2, \dots) = \left( \Psi^1(t, x_1, x_2, \dots), \dots, \Psi^d(t, x_1, x_2, \dots) \right)$ ,  $x_i \in \Omega \subset \mathbf{R}^6$ ,  $n$  is a number of particles:

$$\begin{aligned} L\Psi &\equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \\ Q &\equiv Q_{d_0, d_1, d_2, \dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) \\ &= \sum_{i_0, i_1, i_2, \dots=1}^{d_0, d_1, d_2, \dots} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots) \left( \frac{\partial}{\partial t} \right)^{i_0} \left( \frac{\partial}{\partial x_1} \right)^{i_1} \left( \frac{\partial}{\partial x_2} \right)^{i_2} \dots \int \mu_k. \end{aligned}$$

Let us consider the  $N$  mode approximation:

$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0, i_1, i_2, \dots=1}^N a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots(t, x_1, x_2, \dots).$$

We will determine the expansion coefficients from the following conditions (related to the proper choosing of variational approach) which are nothing but Generalized Dispersion Relations (GDR) :

$$\ell_{k_0, k_1, k_2, \dots}^N \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) dt dx_1 dx_2 \dots = 0. \quad (2)$$

Thus, we have exactly  $dN^n$  algebraical equations for  $dN^n$  unknowns  $a_{i_0, i_1, \dots}$ . This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second one. It allows to unify the multiresolution expansion with variational construction [8]–[22]. As a result, the solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the generic operator  $L$ ) and the rest are linear problems related to the computation of the coefficients of reduced algebraic equations. It is also related to the choice of exact measure of localization (including the class of smoothness), which is proper for our set-up. These coefficients can be found via functional/algebraic methods by using the compactly supported wavelet basis or any other wavelet families [6]–[7]. As a result, the solution of the hierarchies as in c- as in q-region, has the following multiscale or multiresolution decomposition via nonlinear localized eigenmodes (Fig. 1):

$$\begin{aligned} W(t, x_1, x_2, \dots) &= \sum_{(i, j) \in \mathbb{Z}^2} a_{ij} U^i \otimes V^j(t, x_1, \dots), \\ V^j(t) &= V_N^{j, \text{slow}}(t) + \sum_{l \geq N} V_l^j(\omega_l t), \quad \omega_l \sim 2^l, \\ U^i(x_s) &= U_M^{i, \text{slow}}(x_s) + \sum_{m \geq M} U_m^i(k_m^s x_s), \quad k_m^s \sim 2^m, \end{aligned} \quad (3)$$

which corresponds to the full multiresolution expansion in all underlying time/space scales. The formulas (3) give the expansion into a slow part and fast oscillating parts for arbitrary  $N, M$ . So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process. In this way, one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. Representation (3) do not use perturbation techniques or linearization procedures. Numerical calculations are based on compactly supported wavelets and wavelet packets and on the evaluation of accuracy on the level  $N$  of the corresponding cut-off of the full system regarding Fock-like norm described above:

$$\|W^{N+1} - W^N\| \leq \varepsilon.$$

### 3. CONCLUSIONS

By using high localized nonlinear eigenmodes with their best phase space localization properties, we can describe the full zoo of possible complex patterns generated from localized (coherent) structures/orbits in quantum systems with complicated behaviour due to process of quantum self-organization (Figs. 2–8).

The numerical simulation demonstrates the formation of various (meta) stable patterns or orbits generated by internal hidden symmetry from generic high-localized fundamental modes (Fig. 1). These (nonlinear) eigenmodes, definitely, are more realistic for the modeling of classical/quantum dynamical process than infinite smooth linear gaussian-like coherent states. Here we mention only the best convergence properties of the expansions based on wavelet packets, which realize the minimal Shannon entropy property and the exponential control of the convergence of expansions like (3).

Fig. 2 demonstrates results of direct modeling for the non-trivial Wigner function with non-trivial interference picture for three best localized wavelet packets (Fig. 1) [6], [7].

Fig. 5 presents wavelet state defined as a state with minimum entropy and zero measure, which is generated by a finite number of fundamental modes only (more exactly, only a few modes contribute to the energy spectrum). It corresponds to the (possible) result of einselection [4] after decoherence process started from chaotic/entangled-like state (Figs. 4, 6).

Figs. 3, 4 and 7, 8 demonstrate the steps of multiscale resolution from level four to level six, or the degrees of interference, or degree of self-interaction, or intermittency-like behaviour during the quantum interaction/evolution of entangled states or quantum self-organization leading to the growth of the degree of entanglement.

It should be noted that, in addition, we can control the type of behaviour on the level of the reduced algebraic system (Generalized Dispersion Relation) (2). We hope that it will be important in practical applications. Refs. [8]–[23] contains a lot of related methods and approaches in similar complex physical problems.

### 4. SUMMARY AND PERSPECTIVES

In these two Parts we considered some generalization of the theory of quantum states, which is based on the analysis of long standing problems and unsatisfactory situation with possible interpretations of quantum mechanics. We demonstrate that the consideration of quantum states as sheaves can provide, in principle, more deep understanding of some phenomena. The key ingredients of the proposed construction are the families of sections of sheaves with values in the category of the functional realizations of infinite-dimensional Hilbert spaces with special (multiscale) filtration.

The questions we hope to answer are:

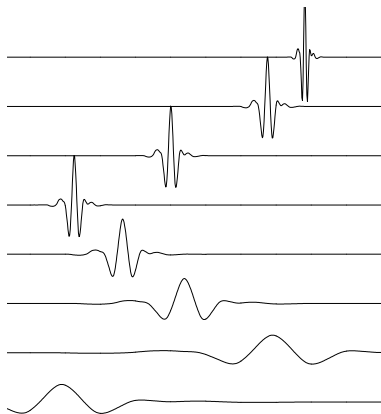


Figure 1. Nonlinear localized basis eigenmodes.

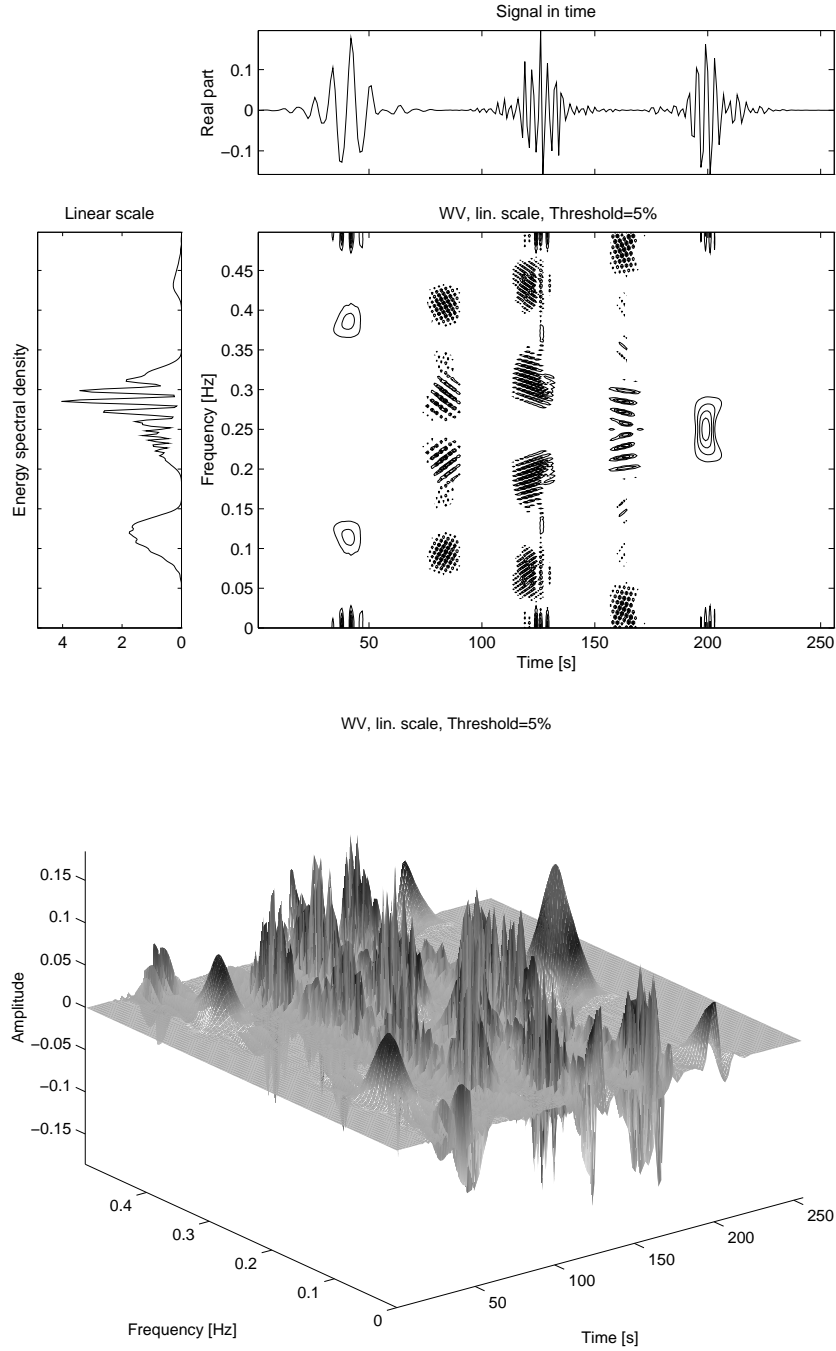


Figure 2. Wigner function for three wavelet packets: direct modeling

- i) How may we enlarge the amount of (physical) information stored in one (quantum) physical point?
- ii) Relation between structureless geometrical points and physical points (or point objects like (point) particles) with rich (possible hidden) structure.
- iii) How we may “resolve” (physical) point/quantum state to provide such a structure.
- iv) A new look in new framework for localization, entanglement, measurement, and all that.

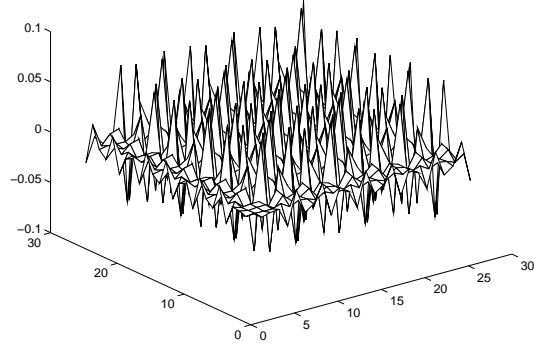


Figure 3. MRA approximation for Wigner function.

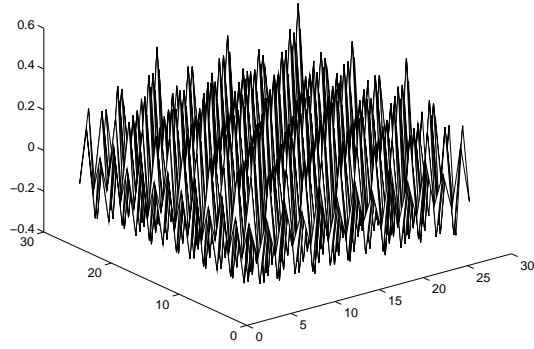


Figure 4. MRA approximation for Wigner function: chaotic-like quantum pattern.

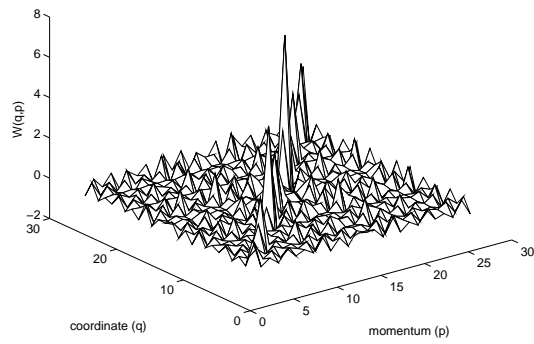


Figure 5. Localized quantum pattern: (wavelet) Wigner function.



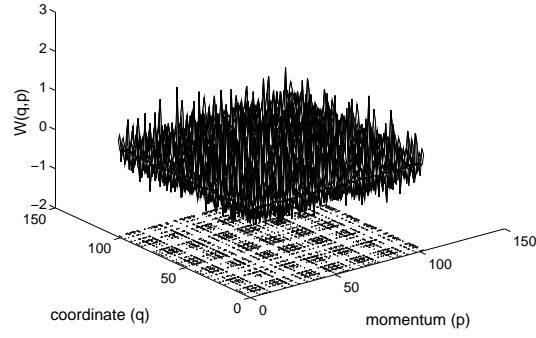


Figure 6. Entangled-like Wigner function.

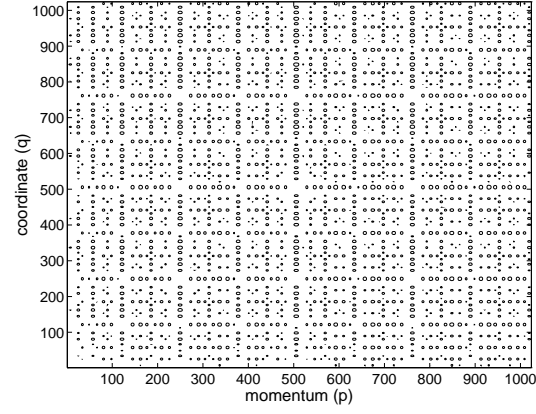


Figure 7. Interference picture at the scale level four: approximation for Wigner function.

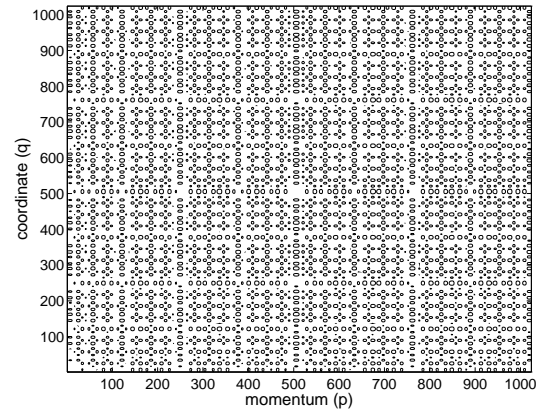


Figure 8. Interference picture at the scale level six: approximation for Wigner function.

- v) How we may to explain/re-interpret a standard zoo of standard interpretations/phenomena (multiverse, wave functions collapse, hidden parameters, Dirac self-interference, ensemble interpretation, etc, etc.) of Quantum Mechanics in the new framework.

The long-range aims of approaches presented in Part 1 [23] are to compare the following key objects which are basic for any type of the exposition of Quantum Mechanics (and other related areas):

*Geometrical points vs. Physical Points (or Point Objects, or One-Point-Patterns);*

*Point functions vs. Sheaves;*

*Partial Differential Equations (and proper orthodox approaches)*

*vs.*

*Pseudodifferential Equations (via Microlocal analysis and all that).*

The more heuristic Part 1 [23] is continued by Part 2 where we make a sketch of technical details and comparison:

*Fourier/Gaussian modes vs. (pretty much) Localized (but non-gaussian) Physical Modes;*

*Fourier Analysis vs. Local Nonlinear Harmonic Multiscale Analysis (including wave- and other -lets and multiresolution);*

*Standard Quantum Images vs. Quantum Patterns generated by Multiresolution;*

Definitely, the final point after unification of the constructions from both Parts is

*Categorification Procedure(s) for Quantum Mechanics (QM).*

It is more or less obvious that we do not have any unified framework covering a zoo of (mostly) discrepant interpretations of QM, as well as satisfactory explanation/understanding of main ingredients of a phenomena like entanglement, etc.

The starting point is an idea to describe (to resolve) the key object of the area, namely (quantum) point objects (quantum physical points or particles, e.g., photons or electrons, structureless at first glance or structuredness at all, in the physical reality).

Usually, for the modeling of real physical point objects, one can consider equivalence between them and standard geometrical points (at the moment, the concrete description of the set to which such points belong as one-element subsets, does not matter).

As direct consequence, our dynamical variables like wave function or density matrix or the Wigner function are described by means of point functions or what mathematicians mean by standard functions. But, as we can understand, a geometrical point is structureless and it seems that we need much more to enlarge the amount of data/information corresponding to this generic object. To advocate this Hypothesis in the present context it is worth noting Dirac's famous sentence "an electron can interact only itself via the process of quantum interference". Roughly speaking, from this perspective it means that a "point particle" needs and must have a non-trivial complicated structure. It seems reasonable to have the rich structure for a model of the (quantum) physical point (usually named as "particle") in comparison with the structureless geometrical point. All above looks like Physical Hypothesis and it may be not so clear but at the same time it is more or less well known from the mathematical point of view if we accept the right (Wigner-Moyal-~~Quantization~~) picture for description of Quantum World. In that framework, more exactly Strict Deformation Quantization approach, all equations are pseudodifferential and as immediate consequence we need to change point functions by sheaves what provides the clear resolution of the (Physical) Point: as a result it is no more structureless but acquires the rich (possible hidden at first glance) structure.

To summarize, our first three Hypotheses are as follows:

### **(Physical) Hypothesis 1**

Physical Point Object (physical point, point particle) is not a structureless object and cannot be described by means of the geometrical point (in the standard math sense). Instead of that, Physical Points have a rich (infinite) hidden structure.

### (Physical version) Hypothesis 2

Quantum Dynamical Variables (wave functions, density matrix, Wigner functions) are not point functions but sheaves defined on the properly chosen space-time manifold (or topological space, or variety, or even scheme).

### (Math version) Hypothesis 2

To provide the structuredness of the Physical Point, allowing to enlarge a number of useful properties and increase the amount of corresponding data inside, we consider it together with a proper generalization of wave function as a section/fiber of proper sheaf (in a proper category of objects) defined on a proper model (category) of space-time.

### (Physical) Hypothesis 3

Deformation Quantization Picture (roughly speaking, Wigner-Weyl-Moyal) is not in contradiction with Hypotheses 1 and 2 and allow us to consider the proper description.

### (Math) Hypothesis 3

Microlocal Analysis of proper Pseudodifferential Dynamical Equations (like the Wigner one) allows to create a model of infinite hierarchy for the correct representation of Physical Point.

In such an approach **Quantum States** are (roughly speaking) sections of the so-called **coherent sheaves** or **contravariant functors** from the proper category corresponding to space-time to other one properly describing the complex dynamics of Quantum States/Patterns. As we sketched in this Part 2, the objects of this category are some filtrations on the proper functional realization of Hilbert space of States. In this picture, a result of **measurement** corresponds to the so called **direct (inductive) limit**.

Definitely, we need a proper analytical/numerical machinery to realize all that. It is exactly the subject of approaches presented in Part 2. It is not worth mentioning that all above is well known in Mathematics long ago. But we hope to convert this knowledge to a physical acceptable and computational form. We present additional details in a separate paper [23].

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